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# Numerical Analysis of Abel Equations and Fractional-Order Chen Systems Using an Adaptive P–C Method

Shahinaz. A. Elsamani<sup>1\*</sup>, Khaled Helmi Khashan<sup>2</sup>

1. Department of Mathematics, Faculty of Science, Bakht Al-Ruda University, Duaim.
  2. Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
- \*Correspondence : [Shahinazel121@gmail.com](mailto:Shahinazel121@gmail.com)

**Abstract:** This study is presented to solve the Abel differential equation in canonical form and the four-dimensional fractional order Chen system under generalized Caputo-type derivatives by applying the adaptive predictor-corrector method, known as the adaptive (P-C) method. Additionally, it conducts a comparative analysis between the proposed technique and the Runge-Kutta Fourth Order (RK4) method. The findings reveal the effectiveness of the proposed approach, generating solutions that closely match the approximate results obtained via the Runge-Kutta Fourth Order (RK4) method. As a result, extending this methodology to a wide array of systems becomes feasible, enhancing the results' precision. Furthermore, this method exhibits utility in accurately identifying instances of attractor chaos through empirical demonstrations. In prospective applications, this method holds promise for numerically solving diverse models relevant to science and engineering.

**Keywords:** The adaptive (P-C) method; Numerical solution; Generalized fractional derivatives; Chaos

**Citation:** Elsamani, S. A & Khashan, K. H. Numerical Analysis of Abel Equations and Fractional-Order Chen Systems Using an Adaptive P–C Method. Central Asian Journal of Mathematical Theory and Computer Sciences 2026, 7(2), 224-237

Received: 10<sup>th</sup> Dec 2025  
Revised: 21<sup>th</sup> Jan 2026  
Accepted: 04<sup>th</sup> Feb 2026  
Published: 31<sup>th</sup> Mar 2026



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## 1. Introduction

Calculus with non-integers as the order is referred to as fractional calculus and is now relevant when compared to traditional integer forms of derivatives and integrals in numerous fields such as physics, engineering, applied mathematics, and more [1][2][3][4][5][6][7][8][9][10][11][12]. It provides a versatile approach to describe phenomena such as diffusion, viscoelasticity, and control systems. Researchers have developed diverse methods to solve fractional differential equations, aided by advancements in computing and software tools [13][14][15][16]. These techniques, rooted in solid mathematical foundations, have applications in physics and engineering and areas like economics, biology, and management [17][18][19][20][21][22][23][24].

Many scientific and engineering sectors use chaotic system modeling articles, a trend that has grown in recent years [25][26][27][28][29][30]. Many of these works address the challenge of incorporating chaotic systems into electrical circuit modeling for chaotic applications. Reliable forecasts are difficult due to the complexity of real-world phenomena and chaotic modeling. We can use phase portraits to analyze the effects of model parameters on system behavior, Lyapunov exponents, and chaotic and hyperchaotic tendencies.

Fractional calculus has been applied in new ways, creating a new field of study. Many fractional operators have appeared in this sector. Caputo derivatives are widely used because they solve real-world problems [31][32]. In addition, fractional operators are using Mittag-Leffler kernels and exponential kernels [33][34][35]. Additionally, the

utilization of the Atangana-Baleanu derivative is touched upon in [36][37], which offers advantages when it comes to modeling situations that occur in the actual world. The evidence shown in [38] demonstrates that there has been a growing interest in investigating the impact that memory modeling has on chaotic and hyperchaotic systems, which has revealed a new opportunity for research. It is essential to bring to your attention that even minute modifications to the initial conditions of chaotic systems can potentially reduce the occurrence of chaotic or hyperchaotic behavior. This section aims to provide an overview of existing publications on the topic. In particular, [39] delves into the chaotic behavior of the Chua circuit. In [40], the authors explore and substantiate the Lyapunov fractional exponent approach. Another alternative formulation is discussed in references [41][42][43][44][45][46]; their primary focus revolves around Caputo derivatives, bifurcations, and Lyapunov analysis. On the contrary, references [47][48] encompass a diverse spectrum of fractional order chaotic and hyperchaotic systems employing Caputo derivatives. Furthermore, amplitude control is instrumental in capturing dynamically symmetric systems, as elucidated in the reference [49].

Fractional integrals enable the creation of various fractional derivatives, including well-known ones like Riemann-Liouville fractional integral (RLFI), Hadamard, and Caputo.

The (RLFI) of order  $\alpha > 0$ , given by:

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t > a. \quad (1)$$

By using Eq. (1), the (CFD) and the (RLD) with  $\alpha > 0$  given by:

$$\begin{aligned} \left( {}^R D_{a+}^{\alpha} f(t) = D^m I_{a+}^{m-\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-\alpha-1} f(s) ds, \quad t > a, \right. \\ \left. {}^C D_{a+}^{\alpha} f(t) = I_{a+}^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds, \quad t > a, \right) \quad (2) \end{aligned}$$

where  $m-1 < \alpha \leq m$  and  $m \in \mathbb{N}$ . See [49].

This research's novelty lies in its innovative application of the adaptive (P-C) method to solve the fractional Abel equation and the 4-D fractional-order Chen system, its comparative analysis with the RK4 method, and its demonstration of the method's effectiveness and utility in identifying chaos. This research can potentially advance numerical methods in science and engineering, specifically in dealing with complex, non-linear systems.

The paper is organized as follows: Section 2 provides basic definitions. Section 3 presents the adaptive (P-C) method. Section 4 discusses the numerical solution of the first kind of Abel differential equation and the 4-D fractional-order Chen system using the adaptive (P-C) scheme, including numerical results. Section 5 concludes the study.

## 2. Materials and Methods

A characteristic feature of this study is the method used, which is based on the approach for the numerical solution of fractional differential equations where an adaptive predictor-corrector (P-C) method has been utilized under generalized Caputo-type derivatives. First, the problems to be solved, i.e., the canonical Abel differential equation and the four-dimensional fractional-order Chen system, are stated as IVPs in the general fractional setting. First, we transform these problems into equivalent integral equations and then we approximate these integral equations numerically. After that, the computational domain is discretized by a non-uniform mesh (single point), where adaptive step-size control can be used which helps to enhance the accuracy and the stability over non-uniform grid solutions. The numerical algorithm starts with a predictor step based on a one-step Adams-Bashforth method to get preliminary approximations of the solution. Then, a trapezoidal quadrature-based corrector stage is applied to adjust the predictions based on the weighted sums of past approximations. This iterative predictor-corrector mechanism guarantees convergence and increases the accuracy of the numerical solution by looping back and forward between the two methods. The approach is implemented in a numerical algorithm and applied to both illustrative problems and numerical results are produced for different sets of fractional orders and parameters. The solutions obtained from the proposed technique is compared to that of classical Runge-Kutta fourth-order (RK4) method to validate the efficiency of the proposed approach.

Also, phase portraits, parametric plots, etc., are graphical simulations that help us analyze the dynamics of a system and identify any chaotic behavior.) It further investigates the effects of fractional parameters on solution tendencies, showing greatly that the generalized Caputo operator offers flexibility in modeling this type of memory dependent systems. So all in all, this systematic numerical method gives a solid and precise basis for resolving complicated landscape nonlinear fractional systems.

### 3. Results and Discussion

#### Basic definitions

This section briefly overviews the operator derivatives, mainly focusing on fractional variants central to our analysis.

**Definition 1.** For continuous functions  $f$ , the generalized (FI) denoted by  $I_{a+}^{\alpha,\rho} f(t)$ ,  $\alpha > 0$ , and  $\rho > 0$ , is given by

$$I_{a+}^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} f(s) ds, \quad \alpha > 0, t > a. \quad (3)$$

For  $m - 1 < \alpha \leq m$  where  $m \in \mathbb{N}$ , see [50].

**Definition 2.** For continuous functions  $f$ , then the generalized (RLFD) denoted by  ${}^R D_{a+}^{\alpha,\rho} f(t)$ , of order  $\alpha > 0$  is given by

$${}^R D_{a+}^{\alpha,\rho} f(t) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)} \left( t^{1-\rho} \frac{d}{dt} \right)^m \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{m-\alpha-1} f(s) ds, \quad t > a \geq 0. \quad (4)$$

**Definition 3.** For continuous functions  $f$ , then the generalized (CFD) denoted by  ${}^C D_{a+}^{\alpha,\rho} f(t)$ , of order  $\alpha > 0$  is given by

$${}^C D_{a+}^{\alpha,\rho} f(t) = \left( {}^R D_{a+}^{\alpha,\rho} \left[ f(x) - \sum_{n=0}^{m-1} \frac{f^{(n)}(a)}{n!} (x - a)^n \right] \right) (t), \quad t > a \geq 0, \quad (5)$$

where  $m = [\alpha]$  and  $\rho > 0$ . In case of  $0 < \alpha \leq 1$ , see [51].

$${}^C D_{a+}^{\alpha,\rho} f(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)} \int_a^t (t^\rho - s^\rho)^{-\alpha} f'(s) ds, \quad 0 < \alpha \leq 1, t > a \geq 0. \quad (6)$$

**Definition 4.** The new generalized (CFD) operator,  $\mathcal{D}_{a+}^{\alpha,\rho}$ ,  $\alpha > 0$  is given by:

$$(\mathcal{D}_{a+}^{\alpha,\rho} f)(t) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)} \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{m-\alpha-1} \left( s^{1-\rho} \frac{d}{ds} \right)^m f(s) ds, \quad t > a, \quad (7)$$

where  $\rho > 0, a \geq 0$ , and  $m - 1 < \alpha < m$ , see [49]

### Algorithm of the adaptive predictor-corrector method

This section presents the adaptive (P-C) method, an algorithm designed for the efficient numerical solution of initial value problems (IVPs) involving the generalized (CFD).

$$\begin{cases} \mathcal{D}_{a+}^{\alpha,\rho} y(t) = f(t, y(t)), t \in [0, T], \\ y^{(k)}(a) = y_0^k, k = 0, 1, \dots, [\alpha], \end{cases} \quad (8)$$

where  $\mathcal{D}_{a+}^{\alpha,\rho}$  is CFD, for  $m - 1 < \alpha \leq m, a \geq 0, \rho > 0$  and  $y \in C^m([a, T])$ , the IVP (7) is equivalent, we get:

$$y(t) = u(t) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} f(s, y(s)) ds, \quad (9)$$

where

$$u(t) = \sum_{n=0}^{m-1} \frac{1}{\rho^n n!} (t^\rho - a^\rho)^n \left[ \left( x^{1-\rho} \frac{d}{dx} \right)^n y(x) \right]_{x=a}. \quad (10)$$

Assuming the function  $f$  has a unique solution on  $[a, T]$ , we partition the interval into  $N$  unequal subintervals  $\{[t_k, t_{k+1}], k = 0, 1, \dots, N - 1\}$  using mesh points.

$$\begin{cases} t_0 = a, \\ t_{k+1} = (t_k^\rho + h)^{1/\rho}, k = 0, 1, \dots, N - 1, \end{cases} \quad (11)$$

where  $h = \frac{T^\rho - a^\rho}{N}$ . Now, to numerically solve the IVP, we will build approximations  $y_k, k = 0, 1, \dots, N$ . If we have previously assessed the approximations  $y(t_j)$  and  $y_j \approx y(t_j) (j = 1, 2, \dots, k)$ , we want to approximate using the integral equation.  $y_{k+1} \approx y(t_{k+1})$ .

$$\begin{aligned} y(t_{k+1}) &= u(t_{k+1}) \\ &+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^{t_{k+1}} s^{\rho-1} (t_{k+1}^\rho - s^\rho)^{\alpha-1} f(s, y(s)) ds. \end{aligned} \quad (12)$$

Making the substitution

$$z = S^\rho, \quad (13)$$

we get

$$\begin{aligned} y(t_{k+1}) &= u(t_{k+1}) \\ &+ \frac{\rho^{-\alpha}}{\Gamma(\alpha)} \int_a^{t_{k+1}^\rho} (t_{k+1}^\rho - z)^{\alpha-1} f(z^{1/\rho}, y(z^{1/\rho})) dz. \end{aligned} \quad (14)$$

That is

$$y(t_{k+1}) = u(t_{k+1}) + \frac{\rho^{-\alpha}}{\Gamma(\alpha)} \sum_{j=0}^k \int_{t_j^\rho}^{t_{j+1}^\rho} (t_{k+1}^\rho - z)^{\alpha-1} f(z^{1/\rho}, y(z^{1/\rho})) dz. \quad (15)$$

Following that, an application of the trapezoidal quadrature method with respect to the weight function  $(t_{k+1}^\rho - \cdot)^{\alpha-1}$ . We obtain the corrector formula for in order to get a close approximation of the right-hand side of Eq. (15), we derive the corrector formula for  $y(t_{k+1}), k = 0, 1, \dots, N - 1$ ,

$$y(t_{k+1}) \approx u(t_{k+1}) + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^k a_{j,k+1} f(t_j, y(t_j)) + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha + 2)} f(t_{k+1}, y(t_{k+1})) \quad (16)$$

where

$$a_{j,k+1} = \begin{cases} k^{\alpha+1} - (k - \alpha)(k + 1)^\alpha & \text{if } j = 0, \\ (k - j + 2)^{\alpha+1} + (k - j)^{\alpha+1} - 2(k - j + 1)^{\alpha+1} & \text{if } 1 \leq j < k \end{cases} \quad (17)$$

The last step of our method is to swap out the value of the predictor,  $y(t_{k+1})$ , obtained by applying the one-step Adams-Bashforth method to the integral equation (14), for the amount  $y^p(t_{k+1})$ . This is accomplished by replacing the function  $f(z^{1/\rho}, y(z^{1/\rho}))$  with the quantity  $f(t_j, y(t_j))$  at each integral in Eq. (16).

$$y^p(t_{k+1}) \approx u(t_{k+1}) + \frac{\rho^{-\alpha}}{\Gamma(\alpha)} \sum_{j=0}^k \int_{t_j^\rho}^{t_{j+1}^\rho} (t_{k+1}^\rho - z)^{\alpha-1} f(t_j, y(t_j)) dz = u(t_{k+1}) + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha + 1)} \sum_{j=0}^k [(k + 1 - j)^\alpha - (k - j)^\alpha] f(t_j, y(t_j)) \quad (18)$$

Hence, the formula fully describes our adaptive (P-C) method for assessing the approximation  $y_{k+1} \approx y(t_{k+1})$ ,

$$y_{k+1} \approx u(t_{k+1}) + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^k a_{j,k+1} f(t_j, y_j) + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha + 2)} f(t_{k+1}, y_{k+1}^p), \quad (19)$$

where  $y_j \approx y(t_j), j = 0, 1, \dots, k$ , and  $y_{k+1}^p \approx y^p(t_{k+1})$  can be determined as described in Eq. (18) with the  $a_{j,k+1}$  in (35). The proposed adaptive (P-C) method uses a non-uniform grid  $\{t_{j+1} = (t_j^\rho + h)^\rho : j = 0, 1, \dots, N - 1\}$ ,  $t_0 = a$  and  $h = \frac{T^\rho - a^\rho}{N}$ , where  $N$  is natural. We cannot use the (P-C) technique to solve IVP defined with the generalized CFD if we employ a uniform grid, as mentioned in [52].

### Applications

This section explores the applicability of the (P-C) approach for the numerical resolution of initial value issues, utilizing the suggested under-generalized Caputo-type derivatives. Considering this, we turned to numerical simulations to investigate potential solutions to our test problems.

**Problem 1.** Let use Consider the first kind of Abel differential equation in canonical form, nonhomogeneous equation with cubic nonlinearity [53]

$$\mathcal{D}_0^{\alpha,\rho} y(t) = \sin t - y(t)^3 \quad (20)$$

This is associated with the condition  $y(0) = 0.5$ .

By using Eq. (19), the approximations  $y_{k+1}$ , and for  $N \in \mathbb{N}$  and  $T > 0$ ,

$$y_{k+1} \approx y_0 + a \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^k a_{j,k+1} (\sin(t_j) - y_j^3) + a \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha+2)} (\sin(t_{k+1}) - y_{k+1}^3) \quad (21)$$

where  $h = \frac{T^\rho}{N}$  and  $y_0 = 0.5$ , then

$$y_{k+1} \approx 0.5 + a \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^k [(k+1-j)^\alpha - (k-j)^\alpha] (\sin(t_j) - y_j^3) \quad (22)$$

The tables display solutions of Equation (20) using the adaptive (P-C) method under different settings to demonstrate the behavior and precision of the numerical technique. **Table 1** presents a comparison of solutions using different value of  $t$ , utilizing various step sizes of  $h$ , in addition to a Runge-Kutta 4th order (RK4) solution as a reference. It can be observed from the table that the solutions gradually reduce as the step size is taken forward. **Table 2** provides the solutions at  $t = 0.1$  for, several  $\alpha, \rho$ . **Table 3** depicts solutions at  $t = 0.2$ . For the same values of the step size and parameter combinations. These results show that the fractional order parameters influence the solution's stability and its convergence.

**Table 1.** Solutions of Equation (20), where  $\alpha = 1, \rho = 1$ .

$h$	$t = 0.1$	$t = 0.2$	$t = 0.5$
1/10	0.49889	0.50233	0.53928
1/20	0.49751	0.49978	0.53459
1/40	0.49686	0.49857	0.53237
1/80	0.49654	0.49798	0.53128
1/160	0.49638	0.49769	0.53075
1/320	0.49630	0.49755	0.53049
RK4	0.49283	0.49575	0.55589

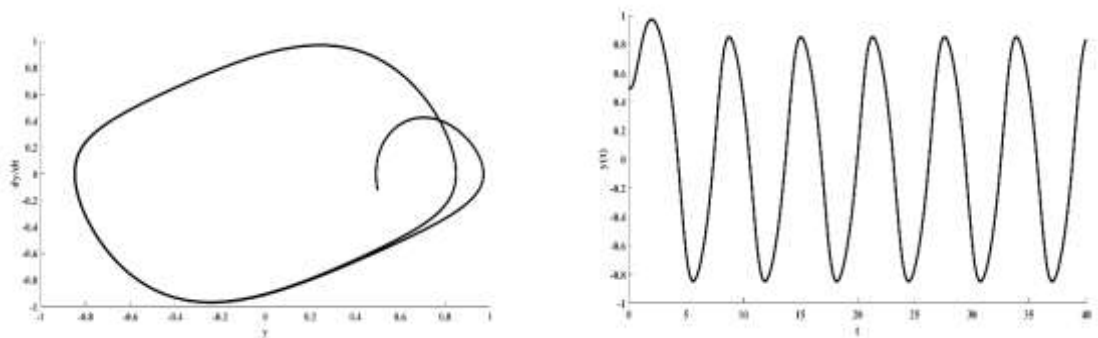
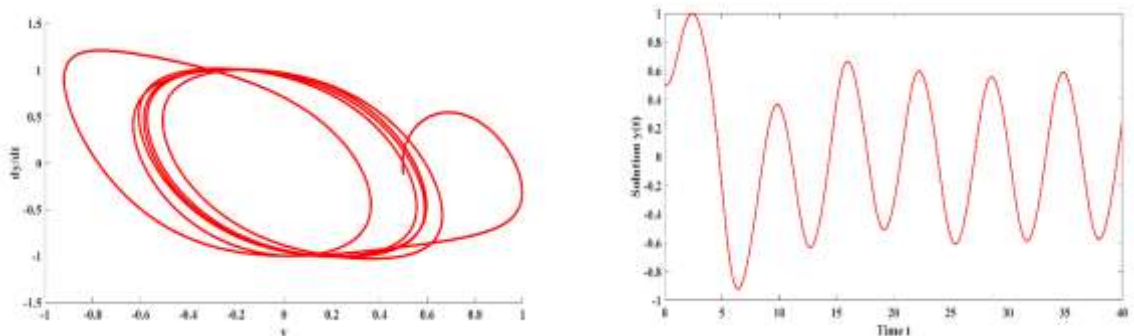
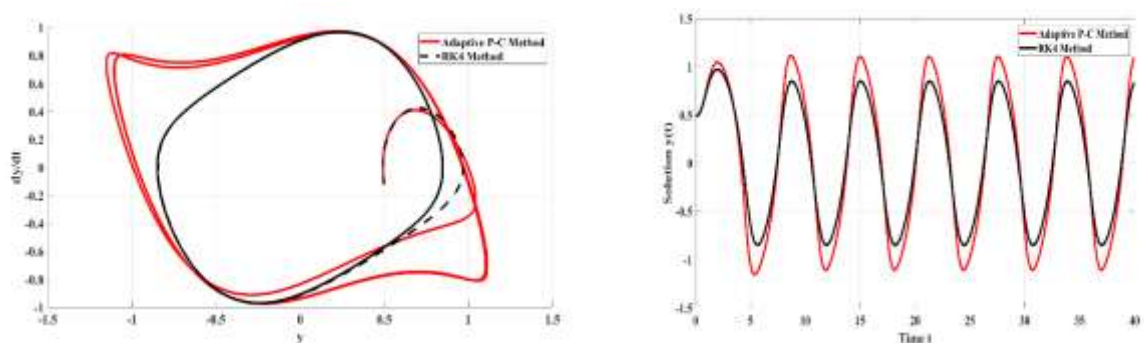
**Table 2.** Solutions of Equation (20), where  $t = 0.1$ .

$h$	$\alpha = 1, \rho = 0.9$	$\alpha = 0.95, \rho = 0.8$	$\alpha = 0.9, \rho = 1.2$
1/10	0.49876	0.49830	0.49856
1/20	0.49724	0.49622	0.49677
1/40	0.49651	0.49509	0.49568
1/80	0.49616	0.49442	0.49493
1/160	0.49598	0.49398	0.49432
1/320	0.49590	0.49363	0.49379
1/640	0.49585	0.49334	0.49327

**Table 3.** Solutions of Equation (20), where  $t = 0.2$ .

$h$	$\alpha = 1, \rho = 0.9$	$\alpha = 0.95, \rho = 0.8$	$\alpha = 0.9, \rho = 1.2$
1/10	0.50259	0.50324	0.50243
1/20	0.49977	0.49962	0.49950
1/40	0.49843	0.49778	0.49788
1/80	0.49778	0.49679	0.49691
1/160	0.49746	0.49621	0.49625
1/320	0.49730	0.49584	0.49574
1/640	0.49722	0.49557	0.49530

**Fig. 1** illustrates the curves and parametric plots obtained using the RK4 method, and the different formal parameters tend to remain stable and periodic over the solution period. These are presented using parametric plots and curves slightly different from those in **Fig. 2** and given the robust nature of the (P-C) method, this shows variations due to the adaptation. Finally, in **Fig. 3**, the graphic shows the proximity of the two methods. The adaptive (P-C) method and the RK4 method nearly coincide, supporting the adaptive method's accuracy in efficiently capturing the system's dynamics.

**Fig. 1:** The curves and parametric plots using RK4.**Fig. 2:** The curves and parametric plots using (P-C) schema when  $(\alpha, \rho) = (1, 1)$ .**Fig. 3:** The curves and parametric plots using RK4, and (P-C) schema when  $(\alpha, \rho) = (0.95, 0.8)$ .

**Problem 2.** Let us consider the following 4-D fractional-order Chen system:

The 4-D fractional-order Chen system was introduced before in [49]. In this part, we present this 4-D fractional system under a generalized Caputo-type fractional derivative operator, which is formulated as a set of Four fractional order differential equations:

$$\begin{aligned} \mathcal{D}_0^{\alpha, \rho} x_1(t) &= a(x_2 - x_1), \\ \mathcal{D}_0^{\alpha, \rho} x_2(t) &= bx_1 - x_1x_3 + cx_2 - x_4, \\ \mathcal{D}_0^{\alpha, \rho} x_3(t) &= x_1x_2 - dx_3, \\ \mathcal{D}_0^{\alpha, \rho} x_4(t) &= x_1 + k, \end{aligned} \quad (23)$$

When  $a = 36, b = -16, c = 28, d = 3,$  and  $k = 0.5,$  with the initial conditions  $x_1(0) = 0, x_2(0) = 0, x_3(0) = 8,$  and  $x_4(0) = 6.$

By using Eq. (19), the approximations  $x_{1,k+1}, x_{2,k+1}, x_{3,k+1}$  and  $x_{4,k+1},$  and for  $N \in \mathbb{N}$  and  $T > 0,$

$$\left\{ \begin{aligned} x_{1,k+1} &\approx x_{10} + a \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^k a_{j,k+1} (a(x_{2j} - x_{1j})) + a \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha + 2)} (a(x_{2k+1}^P - x_{1k+1}^P)), \\ x_{2,k+1} &\approx x_{20} + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^k a_{j,k+1} ((bx_{1j} - x_{1j}x_{3j} + cx_{2j} - x_{4j})) \\ &\quad + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha + 2)} ((bx_{1k+1}^P - x_{1k+1}^P x_{3k+1}^P + cx_{2k+1}^P - x_{4k+1}^P)), \\ x_{3,k+1} &\approx x_{30} + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^k a_{j,k+1} (x_{1j}x_{2j} - dx_3) + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha + 2)} (x_{1k+1}^P x_{2k+1}^P - bx_{3k+1}^P), \\ x_{4,k+1} &\approx x_{30} + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^k a_{j,k+1} (x_{1j} + k) + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha + 2)} (x_{1k+1}^P + k), \end{aligned} \right. \quad (24)$$

where  $h = \frac{T^\rho}{N}$  and Take value of initial condition

$$\left\{ \begin{aligned} x_{1,k+1} &\approx a \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha + 1)} \sum_{j=0}^k [(k+1-j)^\alpha - (k-j)^\alpha] (a(x_{2j} - x_{1j})), \\ x_{2,k+1} &\approx \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha + 1)} \sum_{j=0}^k [(k+1-j)^\alpha - (k-j)^\alpha] (bx_{1j} - x_{1j}x_{3j} + cx_{2j} - x_{4j}), \\ x_{3,k+1} &\approx 8 + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha + 1)} \sum_{j=0}^k [(k+1-j)^\alpha - (k-j)^\alpha] (x_{1j}x_{2j} - dx_3). \\ x_{4,k+1} &\approx 6 + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha + 1)} \sum_{j=0}^k [(k+1-j)^\alpha - (k-j)^\alpha] (x_{1j} + k). \end{aligned} \right. \quad (25)$$

In **Table 4,** we provide numerical solution using the adaptive (P-C) method to Eq. (20) when  $\alpha = 1, \rho = 1$  and  $t = 0.1.$  In **Table 5,** we provide the numerical solution for the value of  $\alpha = 0.95, \rho = 1$  and  $t = 0.5.$

**Table 4.** Solutions of Equation (20) where  $\alpha = 1, \rho = 1$  and  $t = 0.1.$

$h$	$x_1$	$x_2$	$x_3$	$x_4$
1/320	-0.290383800539753	-0.482682680377907	0.372911251720578	1.540986650554822
1/640	-0.291327392292481	-0.484042362875248	0.372962113085596	1.540932864723662
1/1280	-0.291801151497949	-0.484725085283030	0.372987634994008	1.540905854519549

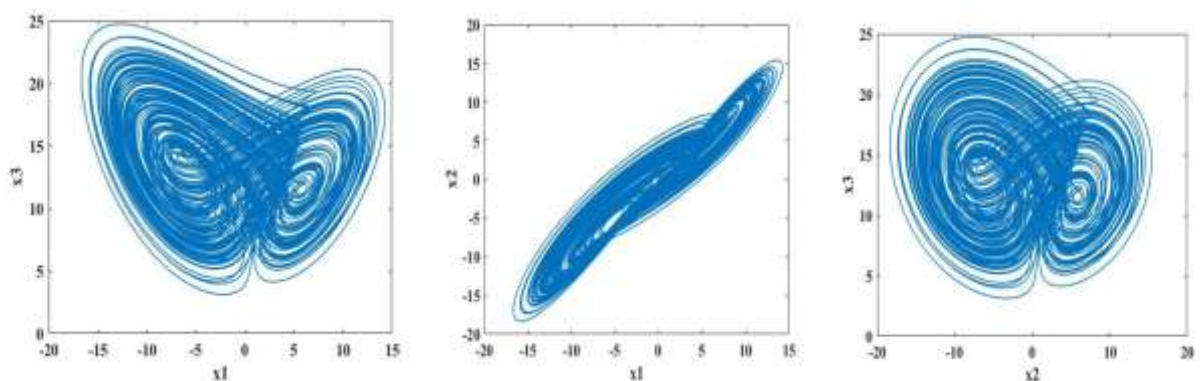
1/2560	-0.292038524362938	-0.485067170346217	0.373000418930677	1.540892319949718
1/5120	-0.292157334416355	-0.485238394291177	0.373006816666778	1.540885545279558
1/10240	-0.292216770386565	-0.485324051673215	0.373010016979550	1.540882156095873
RK4	-0.292276248111265	-0.485409765969730	0.373013217709624	1.540878764789814

**Table 5.** Solutions of Equation (20), where  $\alpha = 0.95$ ,  $\rho = 1$  and  $t = 0.5$ .

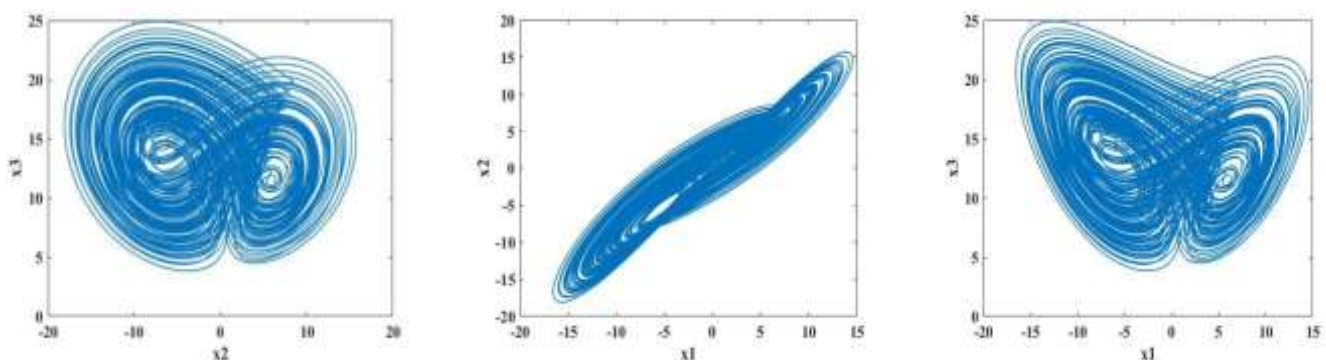
$h$	$x_1$	$x_2$	$x_3$	$x_4$
1/320	12.91566372689926	11.99647046602974	29.660523752565922	-0.313358738792677
1/640	12.775342842142614	11.890966772963202	29.429239821143256	-0.311126766936223
1/1280	12.706205306395653	11.839378452586601	29.314701361057452	-0.310048935164824
1/2560	12.671888900500582	11.813867714337027	29.257706939510236	-0.309519399575302
1/5120	12.654793368173140	11.801182286048045	29.229278290285635	-0.309256957404100
1/10240	12.646261217234890	11.794856943881584	29.215081088064132	-0.309126315313453
RK4	12.647198149462893	11.810267584473269	29.206265207636815	-0.311290297211223

We examine several phase pictures to examine the effect of the generalized Caputo-type fractional derivative on how our fractional system behaves dynamically (Eq. (20)).

We also provide an illustration of a novel technique under the generalized CFD. We investigate the fractional derivative with values for  $\alpha = 0.96, 0.97, 0.98$ , and 1. The development of chaos may be observed with these different orders at  $t = 200$ . Many graphical representations are classified in **Figures (5–8)** show the  $x_1 - x_3, x_1 - x_2$ , and  $x_2 - x_3$  chaotic attractors for fractional orders  $\alpha = 0.96, 0.97, 0.98$ , and 1, respectively.



**Fig. 5:** Chaotic behavior when  $(\alpha, \rho) = (0.96, 0.7)$ .



**Fig. 6:** Chaotic behavior when  $(\alpha, \rho) = (0.97, 0.8)$ .

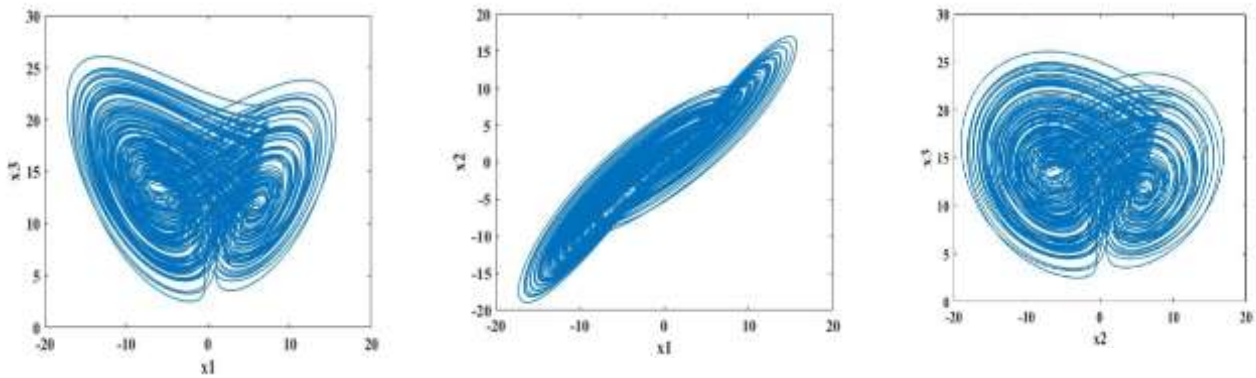


Fig. 7: Chaotic behavior when  $(\alpha, \rho) = (0.98, 0.9)$ .

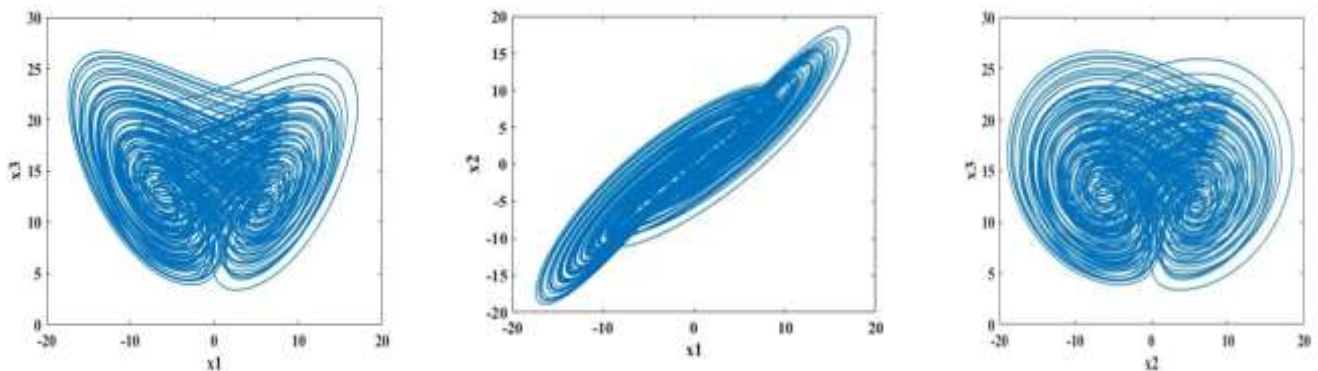


Fig. 8: Chaotic behavior when  $(\alpha, \rho) = (1, 1)$ .

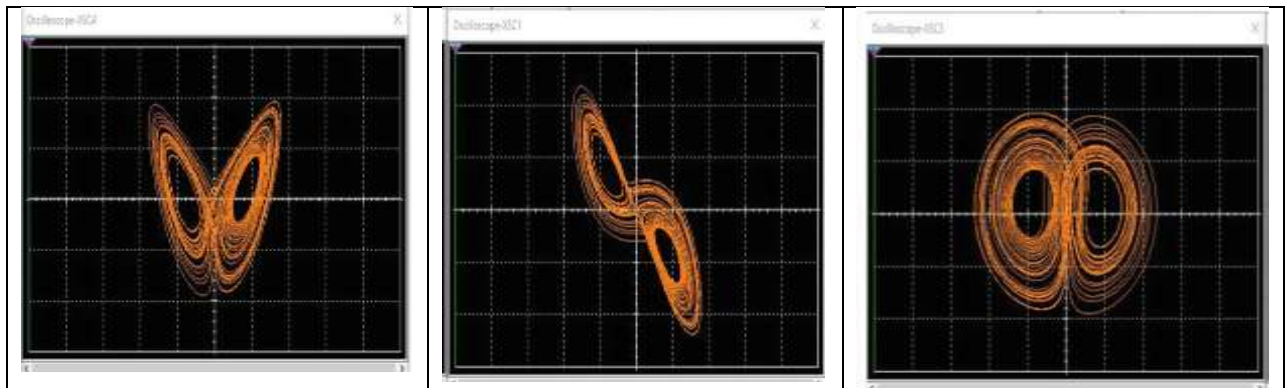


Fig. 9: The circuit generated results when  $(\alpha, \rho) = (1, 1)$ .

In **Figures (5–8)**, we plot numerical solutions to Eq. (20)  $(a, b, c, d, k) = (36, -16, 28, 3, 0.5)$ , with the initial conditions  $x_1(0) = 0, x_2(0) = 0, x_3(0) = 8$ , and  $x_4(0) = 6$ . In Fig 5, we show the circuit generated results for Eq. (20), when  $(\alpha, \rho) = (1, 1)$ .

We display these figures using the adaptive (P-C) method when  $T = 50$  and  $N = 1000$  for some different values of the parameters  $\alpha$  and  $\rho$  in Eq. (20). Furthermore, the importance of changing the  $\alpha$  and  $\rho$  parameters in generalized Caputo fractional models lies in their capacity to capture a broad spectrum of system behaviors, adapt the model to specific applications, and facilitate model validation and optimization. This flexibility is valuable when dealing with complex and diverse real-world systems. Comparing the circuit simulation in **Fig. 9** to numerical findings showed that they were in good agreement.

Chaos theory has wide-ranging applications, from weather forecasting (where small changes in initial conditions can lead to drastically different weather patterns) to studying turbulent fluid dynamics, the behavior of financial markets, and even biological systems like the human heart's rhythm. It is worth noting that while chaotic systems are deterministic and governed by mathematical equations, they can still appear random and unpredictable over extended periods, making them a fascinating area of study and posing challenges in various fields.

Chaos theory has deepened our understanding of the inherent complexity and unpredictability present in many natural and man-made systems. Refer to [54][55][56][57] for further insights. Chaos can manifest in a variety of real-world systems, including coronary arteries within blood vessels [58] and even within cancer and tumor cells [59].

#### 4. Conclusion

This research has centered on the resolution of the Abel differential equation in canonical form and the four-dimensional fractional order Chen system utilizing the adaptive (P-C) method. Additionally, the research has included a detailed comparative analysis, which compares the suggested method with the RK4 method. Our findings underscore the efficacy of the proposed approach, as it consistently yields solutions that closely approximate the results produced by the RK4 method. We provided a numerical strategy using MATLAB software package tools that aided the RK4 method in comparing the solutions. This compatibility with a widely recognized method paves the way for the broad applicability of the adaptive (P-C) method across a spectrum of systems, offering enhanced precision in various numerical simulations. Moreover, this method demonstrates a remarkable capacity for pinpointing attractor chaos through practical demonstrations, affirming its utility in characterizing complex dynamic behaviors. This approach has great potential for the numerical solution of various models relevant to science and engineering domains, fostering advances in computational methodologies for solving intricate systems.

**Funding:** The Researchers Supporting Project number (RSPD2025R920), King Saud University, Saudi Arabia.

**Acknowledgments:** The authors would like to extend their sincere appreciation to the Researchers Supporting Project number (RSPD2025R920), King Saud University, Saudi Arabia.

**Data Availability Statement:** No data were used to support this study.

**Conflicts of Interest:** The authors declare no conflict of interest.

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